Interference channel capacity region for randomized fixed-composition codes

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Abstract—The randomized fixed-composition codes with optimal decoding error exponents are recently studied in [11], [12] for the finite alphabet interference channel with two transmitter-receiver pairs. In this paper we investigate the capacity region for randomized fixed-composition codes. A complete characterization of the capacity region of the said coding scheme is given. The inner bound is derived by showing the existence of a positive error exponent within the capacity region. A simple universal decoding rule is given. The tight outer bound is derived by extending a technique first developed in [10] for single input output channels to interference channels. It is shown that even with a sophisticated time-sharing scheme among randomized fixed-composition codes, the capacity region of the randomized fixed-composition coding is not bigger than the known Han-Kobayashi [24] capacity region. This suggests that the study of the average behavior of randomized codes are not sufficient in finding new capacity regions.

I. INTRODUCTION

The interference channel is a channel model with multiple input-output pairs that share a common communication channel [23]. The capacity region, within which reliable communication can be achieved for all input-output pairs, has been studied [23], [1], [3], [2]. The most well known capacity region result is given in [24], where the capacity region is studied for both discrete and Gaussian cases. Some recent progress on the capacity region are reported in [14], [19], [22], [5], [13]. However, the capacity regions for general interference channels are still unknown. We focus our investigation on the capacity region for a specific coding scheme: randomized fixed-composition codes for which the error probability is defined as the average error probability over all code books with a certain composition (type). Fixed-composition coding is a useful coding scheme in the investigation of both upper [15] and lower bounds of channel coding error exponents [8] for point to point channel and [21], [20] for multiple access (MAC) channels. Recently in [11] and [12], randomized fixed-composition codes were used to derive a lower bound on the error exponent for discrete memoryless interference channels. A lower bound on the maximum-likelihood decoding error exponent is derived, this is a new attempt in investigating the error exponents for interference channels. The unanswered question is the capacity region of such coding schemes.

We answer the above question by giving a complete characterization of the interference channel capacity region for randomized fixed-composition codes. To prove the achievability of the capacity region, we prove the positivity of an achievable error exponent everywhere inside the capacity region. This error exponent is derived by using the method of types [7], in particular, the universal decoding scheme used for multiple-access channels [21]. A better error exponent can be achieved by using the more complicated universal decoding rules developed in [20]. But since they both have the same achievable capacity region, we use the simpler scheme in [21]. To prove the converse, that the achievable region matches the outer bound, we extend the technique in [10] for point to point channels to interference channels by using the known capacity region results for multiple-access channels. The result reveals the intimate relations between interference channels and multiple-access channels. With the capacity region for fixed-composition code established, it is evident that this capacity region is a subset of the Han-Kobayashi region [24].

In this paper we focus on the two input-output case and study the discrete memoryless interference channels with transition probability $W_{Z|X,Y}$ and $\tilde{W}_{Z|X,Y}$ respectively as shown in Figure 1. The two channel inputs are $x^n ∈ \mathcal{X}^n$ and $y^n ∈ \mathcal{Y}^n$, outputs are $z^n ∈ \mathcal{Z}^n$ and $\tilde{z}^n ∈ \tilde{\mathcal{Z}}^n$ respectively, where $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are finite sets. We study the basic interference channel where each encoder only has a private message to the corresponding decoder.

The technical proof of this paper is focused on the average behavior of fixed-composition code books. However this fundamental setup can be extended in the following three directions.

- It is obvious that there exists a code book that its decoding error is no bigger than the average decoding error over all code books. Hence the achievability results in this paper guarantee the existence of a of deterministic coding scheme with at least the same error exponents and capacity region. More discussions are in Section II-E.
- The focus of this paper is on the fixed-composition codes with a composition $P$, where $P$ is a distribution on the input alphabet. This code book generation is different from the non-fixed-composition random coding [16] according to distribution $P$. It is well known in the literature that the fixed-composition codes gives better error exponent result in low rate regime for point to point channels [8].

This work was done when he was a postdoctoral researcher with the Hewlett-Packard Laboratories, Palo Alto, CA.
and multiple-access channels [21], [20]. However they have the same achievable rate region. It is the same case for interference channels and hence the capacity region result in this paper applies to the non-fixed-composition random codes.

- Time-sharing is a key element in achieving capacity regions for multi-terminal channels [6]. For instance, for multiple-access channels, simple time-sharing among fixed-composition codes gives the entire capacity region.

We show that our fixed composition codes can be used to build a time-sharing capacity region for the interference channel. More interestingly, we show that the simple time-sharing technique that gives the entire capacity region for multiple-access channels is not enough to get the largest capacity region, a more sophisticated time-sharing scheme is needed. Detailed discussions are in Section IV.

The outline of the paper is as follows. In Section II we first formally define randomized fixed-composition codes and its capacity region and then in Section II-C we present the main result of this paper: the interference channel capacity region for randomized fixed-composition codes in Theorem 1. The proof is later briefly explained in Section III with more details in [4]. Finally in Section IV, we argue that due to the non-convexity of capacity region of the randomized fixed-composition codes. A more sophisticated time-sharing scheme is needed. This shows the necessity of studying the geometry of the code-books for interference channels.

II. RANDOMIZED FIXED-COMPOSITION CODES AND ITS CAPACITY REGION

We first review the definition of randomized fixed-composition code that is studied intensively in previous works [9], [10], [21], [20]. Then the definition of the interference channel capacity region for such codes is introduced. Then we give the main result of this paper: the complete characterization of the capacity region for randomized fixed-composition codes.

A. Randomized fixed-composition codes

A randomized fixed-composition coding system is a uniform distribution on the code books in which only codeword is from the type set with the fixed composition (type).

First we introduce the notion of type set [6]. A type set $T^n(P)$ is a set of all the strings $x^n \in \mathcal{X}^n$ with the same type $P$ where $P$ is a probability distribution [6]. A sequence of type sets $T^n \subseteq \mathcal{X}$ has composition $P_X$ if the types of $T^n$ converges to $P_X$, i.e., $\lim_{n \to \infty} \frac{N(a|T^n)}{n} = P_X(a)$ for all $a \in \mathcal{X}$ that $P_X(a) > 0$, and $N(a|T^n) = 0$ for all $a \in \mathcal{X}$ that $P_X(a) = 0$, where $N(a|T^n)$ is the number of occurrence of $a$ in type $T^n$. We ignore the nuisance of the integer effect and assume that $n P_X(a)$ is an integer for all $a \in \mathcal{X}$ and $n R_x$ and $n R_y$ are also integers.

In this paper, we study the randomized fixed-composition codes, where each code book with all codewords from the fixed composition being chosen with the same probability. Equivalently, over all these code books, a codeword for message $i$ is uniformly i.i.d distributed on the type set $T^n(P_X)$. A formal definition is as follows.

Definition 1: Randomized fixed-composition codes: for a probability distribution $P_X$ on $\mathcal{X}$, a rate $R$ randomized fixed-composition-$P_X$ encoder picks a code book with the as follows. For any length-$n$ fixed-composition-$P_X$ code book $c_X = (x^n(1), x^n(2), ..., x^n(2^n R_x))$, where $x^n(i) \in T^n(P_X)$, $i = 1, 2, ..., 2^n R_x$, and $x^n(i)$ and $x^n(j)$ may not be different.
for \( i \neq j \), the code book \( c_x \) is chosen with probability
\[
\left( \frac{1}{|T^n(P_X)|} \right)^{2nR_x}.
\]

In other words, the choice of the code book is a random variable that is uniformly distributed on the index set of all code books with fixed-composition \( P_X: \{1, 2, 3, \ldots, |T^n(P_X)|\} \). The chosen code book \( c_x \) is shared between the encoder \( X \) and the decoders \( X \) and \( Y \).

The key property of the randomized fixed-composition codes is that for any message subset \( \{i_1, i_2, \ldots, i_l\} \subseteq \{1, 2, \ldots, 2^nR_x\} \), the codewords for these messages are identical independently distributed on the type set \( T^n(P_X) \).

For randomized fixed-composition codes, the average error probability \( P_e(x) \) for \( X \) is the expectation of decoding error over all message, code books and channel behaviors.

\[
P_e(x) = \left( \frac{1}{|T^n(P_X)|} \right)^{2nR_x} \sum_{c_x} \sum_{c_y} \left( \sum_{m_x} \frac{1}{2^{nR_x}} \sum_{m_y} \frac{1}{2^{nR_y}} \sum_{z^n} W_{Z|XY}(z^n|x^n(m_x), y^n(m_y)) \mathbb{I}(\hat{m}_x(z^n) \neq m_x) \right)
\]

where \( x^n(m_x) \) is the codeword of message \( m_x \) in code book \( c_x \), similarly for \( y^n(m_y) \), \( \hat{m}_x(z^n) \) is the decision made by the decoder knowing the code books \( c_x \) and \( c_y \).

B. Randomized fixed-composition coding capacity for interference channels

Given the definitions of randomized fixed-composition coding and the average error probability in (1) for such codes, we can formally define the capacity region for such codes.

**Definition 2:** Capacity region for randomized fixed-composition codes: for a fixed-composition \( P_X \) and \( P_Y \), a rate pair \( (R_x, R_y) \) is said to be achievable for \( X \), if for all \( \delta > 0 \), there exists \( N_\delta < \infty \), s.t. for all \( n > N_\delta \),

\[
P_e^n(R_x, R_y, P_X, P_Y) < \delta \tag{2}
\]

We denote by \( \mathcal{R}_x(P_X, P_Y) \) the closure of the union of the all achievable rate pairs. Similarly we denote by \( \mathcal{R}_y(P_X, P_Y) \) the achievable region for \( Y \), and \( \mathcal{R}_{xy}(P_X, P_Y) \) for \( (X, Y) \) where both decoding errors are small. Obviously

\[
\mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \cap \mathcal{R}_y(P_X, P_Y). \tag{3}
\]

We only need to focus our investigation on \( \mathcal{R}_x(P_X, P_Y) \), then by the obvious symmetry, both \( \mathcal{R}_y(P_X, P_Y) \) and \( \mathcal{R}_{xy}(P_X, P_Y) \) follow.

C. Capacity region of the fixed-composition code, \( \mathcal{R}_x(P_X, P_Y) \), for \( X \)

The main result of this paper is the complete characterization of the randomized fixed-composition capacity region \( \mathcal{R}_x(P_X, P_Y) \) for \( X \), as illustrated in Figure 2. By symmetry, \( \mathcal{R}_y(P_X, P_Y) \) and then \( \mathcal{R}_{xy}(P_X, P_Y) \) follow.

**Theorem 1:** Interference channel capacity region \( \mathcal{R}_x(P_X, P_Y) \) for randomized fixed-composition codes with compositions \( P_X \) and \( P_Y \):

\[
\mathcal{R}_x(P_X, P_Y) = \{(R_x, R_y): 0 \leq R_x < I(X; Z), 0 \leq R_y \} \bigcup \{(R_x, R_y): 0 \leq R_x < I(X; Z|Y), R_x + R_y < I(X; Y; Z)\}, \tag{4}
\]

where the random variables in (4), \( (X, Y, Z) \sim P_X P_Y W_{Z|XY} \). The region \( \mathcal{R}_x(P_X, P_Y) \) is illustrated in Figure 2.

The achievable part of the theorem states that: for a rate pair \((R_x, R_y) \in \mathcal{R}_x(P_X, P_Y)\), the union of Region I and II in Figure 2, for all \( \delta > 0 \), there exists \( N_\delta < \infty \), s.t. for all \( n > N_\delta \), the average error probability (1) for the randomized code from compositions \( P_X \) and \( P_Y \) is smaller than \( \delta \) for \( X \):

\[
P_e^n(R_x, R_y, P_X, P_Y) < \delta
\]

for some decoding rule. Region II is also the multiple-access capacity region for fixed-composition codes \( (P_X, P_Y) \) for channel \( W_{Z|XY} \).

The converse of the theorem states that for any rate pair \((R_x, R_y) \) outside of \( \mathcal{R}_x(P_X, P_Y) \), that is region III, IV and V in Figure 2, there exists \( \delta > 0 \), such that for all \( n \),

\[
P_e^n(R_x, R_y, P_X, P_Y) > \delta
\]

no matter what decoding rule is used. The definition of the error probability \( P_e^n(R_x, R_y, P_X, P_Y) \) is average over all code books and channel realizations as defined in (1).

The sketch of the proof of Theorem 1 is in Section III with details in [4].

There are two important observations here. First, the capacity region achieved for \( x \) and \( y \) defined as \( \mathcal{R}_{xy}(P_X, P_Y) = \mathcal{R}_x(P_X, P_Y) \cap \mathcal{R}_y(P_X, P_Y) \) is a subset of the capacity region proposed by Han and Kobayashi in [24] which is convex. Hence the randomized fixed-composition codes do not give a bigger capacity region than that in [24]. Secondly, the converse of the randomized coding does not guarantee that there is not a single good fixed-composition code book. The converse claims that, the average (over all code books with the fixed composition) decoding error probability does not converge to zero if the rate pair is outside the capacity region in Theorem 1.

The significance of the above two points is that we cannot hope to get a bigger capacity region than that in [24] by using the fixed-composition random coding scheme which is
sufficient to achieve the entire capacity regions for point to point channel [9], multiple access channels [18] and degraded broadcast channels [17]. This confirms that interference channel coding is a more difficult problem.

D. Necessity of more sophisticated time-sharing schemes

In the achievability part of Theorem 1, we prove that the average error probability for \( X \) is arbitrarily small for randomized fixed-composition codes if the rate pair \((R_x, R_y)\) is inside the capacity region \( R_x(P_X, P_Y) \). For interference channels, it is obvious that the rate region for both \( X \) and \( Y \) is:

\[
R_{xy}(P_X, P_Y) = R_x(P_X, P_Y) \cap R_y(P_X, P_Y),
\]

where \( R_y(P_X, P_Y) \) is defined in the same manner as \( R_x(P_X, P_Y) \) but the channel is \( W_{Z|XY} \) instead of \( W_{Z|X} \) as shown in Figure 1. A typical capacity region \( R_{xy}(P_X, P_Y) \) is shown in Figure 3. It is not necessarily convex.

However, by a simple time-sharing between different rate pairs for the same composition, we can convexify the capacity region. Then the convex hull of the union of all such capacity regions of different compositions gives a bigger convex achievable capacity region. This capacity region of the interference channel is

\[
\text{CONVEX} \left( \bigcup_{P_X, P_Y} R_{xy}(P_X, P_Y) \right).
\]

It is tempting to claim that the above convex capacity region is the largest one can get by time-sharing the “basic” fixed-composition codes as multiple-access channels shown in [6]. However, as will be discussed later in Section IV, it is not the case. A more sophisticated time-sharing gives a bigger capacity region.

This is an important difference between interference channel coding and multiple-access channel coding because the fixed-composition capacity region is convex for the latter and hence the simple time-sharing gives the biggest(entire) capacity region [6]. Time-sharing capacity is detailed in Section IV.

E. Existence of a good code for an interference channel

In this paper we focus our study on the average (over all messages) error probability over all code books with the same composition. For a rate pair \((R_x, R_y)\), if the average error probability for \( X \) is smaller than \( \delta \), then obviously there exists a code book such that the error probability is smaller than \( \delta \) for \( X \). This should be clear from the definition of error probability \( P_{e(x)}(R_x, R_y, P_X, P_Y) \) in (1). In the following example, we illustrate that this is also the case for decoding error for both \( X \) and \( Y \). We claim without proof that this is also true for “uniform” time-sharing coding schemes later discussed in Section IV. The existence of a code book pair that achieves the error exponents in the achievability part of the proof of Theorem 1 can also be shown. The proof is similar to that in [16] and Exercise 30 (b) on page 198 [9].

Similar to the error probability for \( X \) defined in (1), we define the average joint error probability for \( X \) and \( Y \) as
Together with (7), we know that there exists at least one channel capacity region where decoding errors for both users are negligible. For a rate pair \((R_x, R_y) \in \mathcal{R}_{xy}(P_X, P_Y)\), we know that for all \(\delta > 0\), there exists \(N_\delta < \infty\), s.t. for all \(n > N_\delta\), the average error probability is smaller than \(\delta\) for user \(X\) and user \(Y\): 

\[
P^n_{\epsilon(x,y)}(R_x, R_y, P_X, P_Y) < \delta \quad \text{and} \quad P^n_{\epsilon(y)}(R_x, R_y, P_X, P_Y) < \delta.
\]

It is easy to see that the average joint error probability for user \(X\) and \(Y\) can be bounded by:

\[
P^n_{\epsilon(x,y)}(R_x, R_y, P_X, P_Y) = P^n_{\epsilon(x)}(R_x, R_y, P_X, P_Y) + P^n_{\epsilon(y)}(R_x, R_y, P_X, P_Y) \leq 2\delta
\]

From (6), we know that \(P^n_{\epsilon(x,y)}(R_x, R_y, P_X, P_Y)\) is the average error probability of all \((P_X, P_Y)\)-fixed-composition codes. Together with (7), we know that there exists at least one code pair such that the error probability is no bigger than \(2\delta\).

III. SCHEME OF THE PROOF OF THEOREM 1

There are two parts of the theorem, achievability and converse. The achievability part is proved by showing that by using a maximum mutual information decoding rule, positive error exponents exist everywhere in the capacity region in Theorem 1. We apply a method of types argument that is well known for randomized fixed-composition code in the point to point channel coding [9] and MAC channel coding [21]. The converse is proved by giving a non-vanishing lower bound on the error probability outside the capacity region defined in Theorem 1. In the proof, we extended the technique first developed in [10] for point to point channels to interference channels. Due to the page limit, we ignore the proof here. Details are in [4].

IV. DISCUSSIONS ON TIME-SHARING

The main result of this paper is the randomized fixed-composition coding capacity region for \(X\) that is \(\mathcal{R}_x(P_X, P_Y)\) shown in Figure 2. So obviously, the interference channel capacity region, where decoding errors for both \(X\) and \(Y\) are small, is the intersection of \(\mathcal{R}_x(P_X, P_Y)\) and \(\mathcal{R}_y(P_X, P_Y)\) where \(\mathcal{R}_x\) and \(\mathcal{R}_y\) are defined in the similar way but with channel \(W_{Z|XY}\) instead of \(W_{Z|X}\). The intersected region defined in (5), \(\mathcal{R}_{xy}(P_X, P_Y)\), is in general non-convex as shown in Figure 3. Similar to multiple-access channels capacity region, studied in Chapter 15.3 [6], we use this capacity region \(\mathcal{R}_{xy}(P_X, P_Y)\) as the building blocks to generate larger capacity regions.

A. A digression to MAC channel capacity region

Before giving the time-sharing results for interference channels and show why the simple time-sharing idea works for MAC channels but not for interference channels, we first look at \(\mathcal{R}_x(P_X, P_Y)\) in Figure 2. Region II is obviously the multiple access channel \(W_{Z|XY}\) region achieved by input composition \((P_X, P_Y)\) at the two encoders, denoted by \(\mathcal{R}_{mac}^{\text{mac}}(P_X \times P_Y)\). In [6], the full description of the MAC channel capacity region is given in two different manners:

\[
\text{CONVEX} \left( \bigcup_{P_X, P_Y} \mathcal{R}_{xy}^{\text{mac}}(P_X \times P_Y) \right)
\]

\[
\text{CLOSURE} \left( \bigcup_{P_{U}, P_{X|U}, P_{Y|U}} \mathcal{R}_{xy}^{\text{mac}}(P_X \times P_Y | U \times P_U) \right)
\]

where \(\mathcal{R}_{xy}^{\text{mac}}(P_X \times P_Y | U \times P_U) \subset \{ (R_x, R_y) : R_x \leq I(X; Z|Y,U), R_y \leq I(Y; Z|X,U), R_x + R_y \leq I(X,Y; Z|U) \} \) and \(U\) is the time-sharing auxiliary random variable and \(|U| = 4\).

The LHS of (8) is the convex hull of all the fixed-composition MAC channel capacity regions. The RHS of (8) is the closure (without convexification) of all the time-sharing MAC capacity regions. The equivalence in (8) is non-trivial, it is not a consequence of the tightness of the achievable region. It hinges on the convexity of the “basic” capacity regions \(\mathcal{R}_{xy}^{\text{mac}}(P_X, P_Y)\). As will be shown in Section IV-C, this is not the case for interference channels, i.e. (8) does not hold anymore.

B. Simple time-sharing capacity region and error exponent

The simple idea of time-sharing is well studied for multi-user channel coding, broadcast channel coding. Whenever there are two operational points \((R_1, R_2), (R_3, R_4)\), while there exist two coding schemes to achieve small error probability at each operational point, one can use \(\lambda\) amount of channel uses at \((R_1, R_2)\) with coding scheme 1 and \((1-\lambda)\) amount of channel uses at \((R_3, R_4)\) with coding scheme 2. The rate of this coding scheme is \((\alpha R_1 + (1-\alpha) R_2)\) and the error probability is still small

\(3\)The error exponent is, however, at most half of the individual error exponent.
uniformly distributed in \( X \) the decoder, the decoder uses a maximum mutual information
any message coding scheme. The proofs are omitted since they are similar
exponents and lastly drive the achievable rate region for such
time-sharing coding scheme, then give the achievable error
the fixed-composition coding. We first introduce the “uniform”
is not the issue here, the real difference is the intersection
intersections of pairs of non-convex regions (convex or not
region, this is unique to the interference channels, due to
bigger
get better error exponents for MAC channels. This type of
C. Beyond simple time-sharing: “Uniform” time-sharing
In this section we give a time-sharing coding scheme that
we ignore the nuisance of the non-integers here.

\[ \text{Definition 3: “Uniform” time-sharing codes: for a probabil-
ity distribution } P_U \text{ on } \mathcal{U}, \text{ where } \mathcal{U} = \{u_1, u_2, \ldots, u_K\} \text{ with }
\sum_{i=1}^K P_U(u_i) = 1, \text{ and a pair of conditional independent distributions } P_{X|U}, P_{Y|U} \text{. We define the two codeword sets as}
\]

\[ X_c(n) = \{ x^n : x_1^n P_U(u_i) \in P_{X|u_i}, \ldots, x_{n_{P_U(u_i)+1}}^n P_U(u_i) \} \]
\[ Y_c(n) = \{ y^n : y_1^n P_U(u_i) \in P_{Y|u_i}, \ldots, y_{n_{P_U(u_i)+1}}^n P_U(u_i) \} \]
\[ \text{i.e. the } i^{\text{th}} \text{ chunk of the codeword } x^n \text{ with length } n P_U(u_i) \text{ has composition } P_{X_{n_{P_U(u_i)+1}}|u_i}, \text{ and similarly}
\]
\[ Y_c(n) = \{ y^n : y_1^n P_U(u_i) \in P_{Y|u_i}, \ldots, y_{n_{P_U(u_i)+1}}^n P_U(u_i) \} \]
\[ \text{A “uniform” time-sharing code } (R_x, R_y, P_U P_{X|U} P_{Y|U}) \text{ encoder picks a code book with the following probability: for any message } m_x \in \{1, 2, \ldots, 2^{n R_x}\}, \text{ the codeword } x^n(m_x) \text{ is uniformly distributed in } X_c(n), \text{ similarly for encoder } Y.
\]

After the code book is randomly generated and revealed to
the decoder, the decoder uses a maximum mutual information
decoding rule. Similar to the fixed-composition coding, the
decoder needs to either decode both messages \( X \) and \( Y \) jointly

or simply treats \( Y \) as noise and decode \( X \) only, depending
on where the rate pairs are in Region I or II, as shown
in Figure 4. The error probability we investigate is again the
average error probability over all messages and code books.

Theorem 2: Interference channel capacity region \( \mathcal{R}_x(P_U P_{X|U} P_{Y|U}) \) for “uniform” time-sharing codes
with composition \( P_U P_{X|U} P_{Y|U} \):

\[
\mathcal{R}_x(P_U P_{X|U} P_{Y|U}) = \left\{ (R_x, R_y) : 0 \leq R_x < I(X; Z|U), 0 \leq R_y \right\}
\]
\[ \bigcup \left\{ (R_x, R_y) : 0 \leq R_x < I(X; Z|Y, U),
R_x + R_y < I(X, Y; Z|U) \right\} \]

where the random variables in (10), \((U, X, Y, Z) \sim P_U P_{X|U} P_{Y|U} W_{Z|X,Y} \). And the interference capacity region
for \( P_U P_{X|U} P_{Y|U} \) is

\[
\mathcal{R}_{xy}(P_U P_{X|U} P_{Y|U}) = \left( \mathcal{R}_x(P_U P_{X|U} P_{Y|U}) \bigcap \mathcal{R}_y(P_U P_{X|U} P_{Y|U}) \right)
\]

The cardinality of \( \mathcal{U} \) is shown to be no bigger than 7 by
using the Carathéodory Theorem similar to that in [6] for the
capacity region for multiple access channels. We ignore the
proof here.

The rate region defined in (10) itself does not give any new
X-capacity regions for \( X \), since both Region I and II in
Figure 4 can be achieved by simple time-sharing of Region
I and II respectively in (4). But for the interference channel
capacity, we argue in the next section that this coding scheme
gives a strictly bigger capacity region than that given by the
simple time-sharing of fixed-composition codes in (9).

The proof of Theorem 2 is similar to that of Theorem 1.
Details are in [4].

D. Why the “uniform” time sharing is needed?

It is well understood in the literature [18], also briefly
discussed in Section IV-B, that the “uniform” time-sharing
fixed-composition coding gives a bigger error exponent than
the simple time-sharing coding does. More interestingly, we
argue that it gives a bigger interference channel capacity
region. First we write down the interference channel capacity
region generated from the basic “uniform” time-sharing fixed-
composition codes:

\[
\text{CONVEX} \left( \bigcup_{P_X|U} \mathcal{R}_{xy}(P_U P_{X|U} P_{Y|U}) \right)
\]

where \( \mathcal{R}_{xy}(P_U P_{X|U} P_{Y|U}) \) is defined in (11) and
\text{CONVEX}(A) is the convex hull (simple time sharing) of
set \( A \).

\( U \) is a time-sharing auxiliary random variable. Unlike the
MAC coding problem, where simple time-sharing of fixed-
composition codes achieve the full capacity region, it is not
guaranteed for interference channels. The reason is the intersection operator in the basic building blocks in (5) and (11) respectively, i.e. the interference nature of the problem\textsuperscript{5}.

Obviously the rate region by simple time sharing of fixed composition codes in (9) is a subset of simple time sharing of the “uniform” time sharing capacity region (12). In the following example, we illustrate why (12) is bigger than (9).

**Example (symmetric interference channels):** Suppose that we have a symmetric interference channel, i.e. the input alphabet sets $\mathcal{X} = \mathcal{Y}$ and output alphabet sets $\mathcal{Z} = \mathcal{Z}$ and for any $x, y \in \mathcal{X} = \mathcal{Y}$ and for any $z \in \mathcal{Z} = \mathcal{Z}$; the transition probabilities $W_{Z|XY}(z|x,y) = W_{Z|XY}(z|y,x)$.

For such interference channels, we know that the capacity regions $\mathcal{R}_x(P_X, P_Y)$ and $\mathcal{R}_y(P_Y, P_X)$ are symmetric along the 45-degree line $R_x = R_y$. That is, for any $P_X, P_Y$, a rate pair $(R_1, R_2) \in \mathcal{R}_x(P_X, P_Y)$ if and only if $(R_2, R_1) \in \mathcal{R}_y(P_Y, P_X)$.

The comparison of simple timesharing capacity region and the more sophisticated time-sharing fixed-composition capacity region for symmetric interference channels are illustrated by a toy example in Figure 5. For a distribution $(P_X, P_Y)$, the achievable region for the fixed-composition codes is illustrated in Figure 5. $\mathcal{R}_x(P_X, P_Y)$ and $\mathcal{R}_y(P_X, P_Y)$ respectively, these are bounded by the red dotted lines and red dash-dotted lines respectively, so the interference capacity region $\mathcal{R}_{xy}(P_X, P_Y)$ is bounded by the pentagon $ABEFO$. By symmetry, $\mathcal{R}_y(P_Y, P_X)$ and $\mathcal{R}_x(P_X, P_Y)$ are bounded by the blue dotted lines and blue dash-dotted lines respectively, the capacity region $\mathcal{R}_{xy}(P_Y, P_X)$ is bounded by the pentagon $HGDCO$. So the convex hull of these two regions is

\textsuperscript{5}To understand why intersection is the difference but not the non-convexity, we consider the following toy example: four convex sets: $A_1, A_2, B_1, B_2$. We show that $\text{CONVEX}(A_1 \cap B_1, A_2 \cap B_2)$ can be strictly smaller than $\text{CONVEX}(A_1, A_2) \cap \text{CONVEX}(B_1, B_2)$. Let $A_1 = B_2 \subset B_1 = A_2$, then $\text{CONVEX}(A_1 \cap B_1, A_2 \cap B_2) = A_1$ is strictly smaller than $\text{CONVEX}(A_1, A_2) \cap \text{CONVEX}(B_1, B_2) = A_2$. ABDCO.

Now consider the following timesharing fixed-composition coding $P_{X|U}P_{Y|U}P_U$ where $U = \{0, 1\}$, $P_U(0) = P_U(1) = 0.5$ and $P_{X|0} = P_{Y|1} = P_X$, $P_{X|1} = P_{Y|0} = P_Y$. The interference capacity region is obviously bounded by the black pentagon in Figure 5. This toy example shows why (12) is bigger than (9).

**V. Future directions**

The most interesting issue of interference channels is the geometry of the two code books. For point to point channel coding, the codewords in the optimal code book is uniformly distributed on a sphere of the optimal compositions and the optimal composition achieves the capacity. For multiple access channels, a simple time-sharing among different fixed-composition codes is sufficient to achieve the whole capacity region, where for each fixed-composition codes, the codewords are uniformly distributed. To get the biggest possible
achievable rate region for interference channels, however, as illustrated in Section IV, a more sophisticated “uniform” time sharing is needed. So what is time sharing? Both simple time sharing and “uniform” time sharing change the geometry of the code books, however, in different ways. Simple time sharing “glue” segments of codewords together due to the independence of the coding in different segments of the channel uses, meanwhile for “uniform” time sharing, codewords still have equal distances between one another. Better understanding of the geometry of code books will help us better understand the interference channels. In this paper, we give a tight outer bound to a class of coding schemes, the time-sharing fixed-composition code. An important future direction is to categorize the coding schemes for interference channels and more outer bound result may follow. This is in contrast to the traditional outer bound derivations [3] where genies are used.

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